Perturbative evaluation of Kolmogorov constant in a self-consistent model of fluid turbulence

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The response integrals of the almost Markovian-Galelian invariant test-field model (TFM) of Kraichnan, generalized to d dimensions, are analyzed. They are found to be both ultraviolet and infrared finite in the range $0<y<6$, the force correlation being $\sim k^{-d+4-y}$ in the wave-number space. The ultraviolet and infrared poles, occurring, respectively, at $y=0$ and $y=6$, are extracted by means of Laurent expansions about these values of y, yielding the Kolmogorov constant both in three and two dimensions: $C_{3\mathrm{D}}=1.64$ and $C_{2\mathrm{D}}=8.097$. These values are remarkably close to the respective renormalization group (RG) results. However, unlike RG, the perturbative TFM results are found to be (approximately) equally sensitive to *both* (ultraviolet and infrared) poles in both three and two dimensions.

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I. INTRODUCTION

It is well known that the spectrum for the conservative cascade of energy in a turbulent fluid takes the universal Kolmogorov form [1,2]

$$E(k)=C\,\overline{\varepsilon}^{2/3}k^{-5/3} \tag{1}$$

(neglecting intermittency corrections), where k is the wave number, $\overline{\varepsilon}$ is the energy injection rate, and C is the universal Kolmogorov constant. This spectrum was obtained by Kolmogorov [1] for three-dimensional turbulence on dimensional arguments where a direct energy cascade, from large to small scales, takes place. Kraichnan conjectured the validity of this spectrum in two-dimensional turbulence also where an inverse energy cascade, from small to large scales, must take place [3,4]. Absolute equilibirium ensemble in two dimensions suggests the possibility of negative-temperature states with a pile up of energy toward a large scale [3]. The mean-square vorticity being a conserved quantity in two dimensions, the energy dissipation rate approaches zero in the limit of zero viscosity, hence excluding the possibility of direct cascade of energy [4]. Various developments in the phenomenologies and ''microscopic'' formulations of the theories of two- and three-dimensional turbulences have been reviewed in Refs. [5–8].

Kraichnan formulated the well-known direct-interaction approximation (DIA) [9,6], a theory of turbulence which resembles the Dyson-Schwinger formulation of quantum field theory (QFT) [10]. A diagrammatic expansion based on the Navier-Stokes equation and subsequent renormalization by resummation, as shown by Wyld [11], makes the analogy with QFT even more transparent.

Unfortunately, the DIA has the difficulty of being self-consistent for the Kolmogorov $k^{-5/3}$ spectrum believed to be true for real turbulence (excluding minor intermittency corrections). Kraichnan identified this failure with the sweeping of smaller eddies by the larger ones, while Edwards [12] identified it with a divergence in the response integral coming from the large scales of motion. To systematically eliminate the spurious effect of sweeping, Kraichnan reformulated the theory in a Lagrangian frame work [13], which indeed was found to be consistent with the Kolmogorov spectrum. This confirms that the failure of the direct-interaction approximation was indeed associated with the spurious sweeping.

The Lagrangian formalism being too cumbersome, Kraichnan considered the Eulerian model problem of the advection of a general vector field, called the test field, by the purely solenoidal velocity field of fluid motion [14,15]. This yielded the dynamics of the solenoidal and compressive parts of the test-field separately. On removing the self-advection terms in these equations, and giving a DIA-like treatment together with Markovianization of the equations, Kraichnan obtained a theory which is self-consistent for the Kolmogorov spectrum (and, in fact, also for the Kraichnan-Batchelor spectrum for the enstrophy cascade in two-dimensional turbulence). He also calculated numerically the Kolmogorov constant both in three and two dimensions [15].

In this paper we analyze the analytic properties of the response integrals of Kraichnan's Galelian invariant self-consistent model (test-field model, abbreviated TFM) generalized to d dimensions and subsequently perform perturbative evaluation of these integrals. Such integrals occur in critical dynamics [16] of systems as varied as liquid helium, antiferromagnet, Heisenberg ferromagnet, etc., and perturbative methods of evaluations of the integrals occurring in critical dynamics have been considered by Bhattacharjee and Ferrell [17]. In the case of Kraichnan's test-field model of turbulence, we find that the response integrals are both ultraviolet and infrared finite in the region $0<y<6$, where the parameter y comes from the force correlation $\sim k^{-d+4-y}$ in the wave-number space. Subsequently, we evaluate these integrals perturbatively by extracting the ultraviolet and infrared poles, occurring, respectively, at $y=0$ and $y=6$ by means of Laurent expansions about these values of y. This facilitates the evaluations of the Kolmogorov constant both in three and two dimensions, resulting in $C_{3\mathrm{D}}=1.64$ and $C_{2\mathrm{D}}=8.097$. These values are remarkably close to the respective renormalization group (RG) results. However, we

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note that RG and TFM have entirely different starting points. The RG scheme usues the (full) Navier-Stokes dynamics (with random stirring) as the starting point whereas for TFM the starting point is the pressureless advection of a test field where the self-advection terms are subsequently dropped from the dynamics in the construction of the model. We find the most surprising fact that the TFM perturbative results are (approximately) equally sensitive to the ultraviolet (UV) and infrared (IR) poles in both three and two dimensions. This feature of the perturbative TFM is entirely different from the known RG schemes which is an expansion about only one value of y $(=0$ or $6)$. A detailed analysis of the calculations and the result have been presented in the last section (Sec. 5) bringing out various differences between RG and perturbative TFM.

II. KRAICHNAN'S MODEL

In this section we generalize Kraichnan's almost Markovian-Galilean invariant model [14,15] (test-field model) to d space dimensions. This model considers advection of a general vector field $\mathbf{v}(\mathbf{x},t)$ by the purely solenoidal fluid dynamic velocity field $\mathbf{u}(\mathbf{x},t)$, the Fourier transform of which can be written in d space dimensions as

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)v_i(\mathbf{k},t)=-ik_l\int\frac{d^d\mathbf{p}}{(2\pi)^d}u_l(\mathbf{p},t)v_i(\mathbf{q},t)+f_i(\mathbf{k},t), \tag{2}$$

where ν is the viscosity, $\mathbf{q}=\mathbf{k}-\mathbf{p}$, and the steady state is assumed to be supported by the external random forcing field $\mathbf{f}(\mathbf{x},t)$.

Now considering the solenoidal and compressive parts of the field $\mathbf{v}(\mathbf{x},t)$, and dropping the self-advection terms, the respective dynamical equations become

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)v_i^S(\mathbf{k},t)=-ik_lP_{ij}(\mathbf{k})\int\frac{d^d\mathbf{p}}{(2\pi)^d}u_l(\mathbf{p},t)v_j^C(\mathbf{q},t)+f_i^S(\mathbf{k},t), \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)v_i^C(\mathbf{k},t)=-ik_l\Pi_{ij}(\mathbf{k})\int\frac{d^d\mathbf{p}}{(2\pi)^d}u_l(\mathbf{p},t)v_j^S(\mathbf{q},t)+f_i^C(\mathbf{k},t), \tag{4}$$

where the superscripts ^{S} and ^{C} denote the solenoidal and compressive parts, respectively, and $P_{ij}(\mathbf{k})=(\delta_{ij}-k_ik_j/\mathbf{k}^2)$, and $\Pi_{ij}(\mathbf{k})=k_ik_j/\mathbf{k}^2$. Further

$$w_i^S(\mathbf{k},t)=P_{ij}(\mathbf{k})w_j(\mathbf{k},t), \tag{5}$$

$$w_i^C(\mathbf{k},t)=\Pi_{ij}(\mathbf{k})w_j(\mathbf{k},t), \tag{6}$$

for any general field $\mathbf{w}(\mathbf{x},t)$, the Fourier transform of which is $\mathbf{w}(\mathbf{k},t)$.

Now giving the equations a similar treatment as that of the direct-interaction approximation and after Markovianization, one obtains, for steady state

$$\eta^S(k)=\frac{4g^2k^2}{S_d(d-1)^2}\int d^dp\, b^S(k,q,p)\frac{E(p)}{\eta^S(k)+\eta^C(q)+\eta(p)}, \tag{7}$$

$$\eta^C(k)=\frac{2g^2k^2}{S_d(d-1)}\int d^dp\, b^C(k,q,p)\frac{E(p)}{\eta^C(k)+\eta^S(q)+\eta(p)}, \tag{8}$$

in d space dimensions, where $\eta^S(k)$, $\eta^C(k)$, and $\eta(k)$ are the respective relaxation rates, and $S_d=2\pi^{d/2}/\Gamma(d/2)$. The scaling factor g was introduced by Kraichnan as the model is equally plausible with respect to scaling the characteristic times. This was evaluated by Kraichnan considering equilibrium situation where the direct-interaction approximation is expected to be exact, yielding $g=1.064$. The geometrical factors in the above integrals (7) and (8) are given by

$$b^S(k,q,p)=\frac{1}{2}b^C(k,q,p)=\frac{1}{2}\left\{1-\left(\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right)^2\right\}\left\{1-\left(\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\right\}. \tag{9}$$

Further, Kraichnan identified the following correlation between the solenoidal part and the actual velocity field:

$$\langle v_i^S(\mathbf{k},t)v_j^S(\mathbf{k}',t')\rangle=\langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle. \tag{10}$$

We assume that the solenoidal velocity field has the correlation

$$\langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle=Q(k,t)P_{ij}(\mathbf{k})(2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')\,\delta(t-t'). \tag{11}$$

We shall further assume that the external driving fields have Gaussian white noise statistics, and the solenoidal part has the correlation

$$\langle f_i^S(\mathbf{k},t)f_j^S(\mathbf{k}',t')\rangle=F^S(k)P_{ij}(\mathbf{k})(2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')\,\delta(t-t'), \tag{12}$$

with

$$F^S(k)=\frac{2D_0}{k^{d-4+y}}, \tag{13}$$

where y is an external parameter. It follows from the above relations that

$$Q(k,0)=\frac{F^S(k)}{2\eta^S(k)}. \tag{14}$$

The energy spectrum $E(k)$, in the steady state, is defined through

$$\frac{1}{2}\langle\mathbf{u}^2(\mathbf{x},t)\rangle=\int_0^\infty E(k)dk, \tag{15}$$

so that it is related to the velocity correlation by

$$E(k)=\frac{d-1}{2}\frac{S_d}{(2\pi)^d}k^{d-1}Q(k,0). \tag{16}$$

With this formulation of the test-field model generalized to d dimensions, we now proceed to find the analytic properties of the response integrals (7) and (8).

III. PROPERTIES OF THE RESPONSE INTEGRALS

Assuming that $\eta(k)$, $\eta^S(k)$, and $\eta^C(k)$ scale like k^z, power counting in Eqs. (7) and (8) gives $z=2-y/3$, after making use of Eqs. (13), (14), and (16). Consequently we get $\eta(k)\sim k^{2-y/3}$ and $E(k)\sim k^{1-2y/3}$. So we take

$$\eta^S(k)=\mu^S\sqrt{C}\overline{\varepsilon}^{1/3}k^{2-y/3}, \tag{17}$$

$$\eta^C(k)=\mu^C\sqrt{C}\overline{\varepsilon}^{1/3}k^{2-y/3}, \tag{18}$$

$$\eta(k)=\mu\sqrt{C}\overline{\varepsilon}^{1/3}k^{2-y/3}, \tag{19}$$

$$E(k)=C\overline{\varepsilon}^{2/3}k^{1-2y/3}. \tag{20}$$

Thus the Kolmogorov spectrum $E(k)\sim k^{-5/3}$ is obtained when $y=4$.

To find the ultraviolet behavior ($p\to\infty$) of the response integrals (7) and (8), we expand the integrand in the limit $p\gg k$, and then pick up the lowest-order contribution in k/p. In this limit we have $q\approx p$, $\eta(p)\approx\eta(q)\gg\eta(k)$ (suppressing the superscripts), and $b^S(k,q,p)\sim 1$. Thus the integral behaves like $p^{-y/3}$ which diverges only for $y\leqslant 0$, when $p\to\infty$. Thus, for the Kolmogorov spectrum which is obtained for $y=4$, the integrals do not pose any problem in the ultraviolet limit.

For the infrared limit ($p\to 0$), we expand the integral in the limit $p\ll k$ and subsequently pick up the lowest contributing order in p/k. Now, $q\approx k$, $\eta(p)\ll\eta(k)\approx\eta(q)$, and $b^S(k,q,p)\sim p^2/k^2$. Thus, the integrals behave like $p^{4-2y/3}$, and hence the integrals diverge for $y\geqslant 6$. Thus we find again that the integral is well behaved on the infrared side for the Kolmogorov case $y=4$.

Thus the response integrals (7) and (8) are both ultraviolet and infrared finite in the region $0<y<6$. Consequently, this situation is unlike the direct-interaction approximation, where the response integral diverges for $y\geqslant 3$, giving rise to a spurious non-Kolmogorov $k^{-3/2}$ spectrum.

IV. PERTURBATIVE EVALUATION

Having analyzed the behavior of the response integrals, we now set out to evaluate the integrals in Eqs. (7) and (8) by means of Laurent expansions about the UV and IR poles at $y=0$ and $y=6$, respectively.

For the UV pole, the integrands are to be Taylor expanded in the limit $p\gg k$. The geometrical factor given by Eq. (9) yields the expansion as

$$b^S(k,q,p)=\frac{1}{2}\left\{1-\left(\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right)^2\right\}^2\left\{1+\frac{2\mathbf{k}\cdot\mathbf{p}}{p^2}+\cdots\right\}. \tag{21}$$

The expansion of the characteristic memory time of the form $\theta_{123}(k,q,p)=[\eta_1(k)+\eta_2(q)+\eta_3(p)]^{-1}$, where the damping factors have the form $\eta_i(k)=c_ik^{2-y/3}$ ($i=1,2,3$) can be obtained as

$$\theta_{123}(k,q,p)=\frac{1}{(c_2+c_3)p^{2-y/3}}\left\{1+\left(2-\frac{y}{3}\right)\frac{c_3}{c_2+c_3}\frac{\mathbf{k}\cdot\mathbf{p}}{p^2}+\cdots\right\}, \tag{22}$$

where the constants c_i can be identified with those appearing in Eqs. (17), (18), and (19). Combining the above two expansions, we get

$$\theta_{123}(k,q,p)b^S(k,q,p)=\frac{1}{2(c_2+c_3)p^{2-y/3}}\left\{1-\left(\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right)^2\right\}^2\times\left[1+\left\{2+\left(2-\frac{y}{3}\right)\frac{c_3}{c_2+c_3}\right\}\frac{\mathbf{k}\cdot\mathbf{p}}{p^2}+\cdots\right]. \tag{23}$$

Thus we observe that the $\mathbf{k}\cdot\mathbf{p}$ term in the square brackets contributes nothing as the angular integration of odd powers of $\mathbf{k}\cdot\mathbf{p}$ vanish. Consequently, the y dependance does not make any difference so far as ϵ expansion (like that in Yakhot-Orszag RG calculations) is concerned.

Now we Taylor expand the integrands in the limit $p\ll k$ in order to obtain the IR pole. This gives

$$\theta_{123}(k,q,p)b^S(k,q,p)=\frac{1}{2(c_1+c_2+c_3)}\frac{p^2}{k^2}\left\{1-\left(\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right)^2\right\}^2+\cdots \tag{24}$$

for $y=6$, for which the integrals are IR marginal.

Using the above expanions from Eqs. (21), (22), (23), and (24) in Eqs. (7) and (8), after properly identifying the constants c_i with those appearing in Eqs. (17), (18), (19), and using Eq. (20) and the following results for angular integrations

$$\oint d\Omega=S_d,\qquad \oint\cos^2\theta\, d\Omega=\frac{S_d}{d},$$

and

$$\oint\cos^4\theta\, d\Omega=\frac{3S_d}{d(d+2)}, \tag{25}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, we finally obtain the Laurent expansion about the poles $y=0$ and $y=6$ as

$$\mu^S = 6g^2 \frac{(d+1)}{d(d-1)(d+2)} \left[\frac{1}{\mu^C + \mu} \left\{ \frac{1}{y} + O(1) \right\} + \frac{1}{\mu^S + \mu^C + \mu} \left\{ \frac{1}{6-y} + O(1) \right\} \right], \tag{26}$$

$$\mu^C = 6g^2 \frac{(d+1)}{d(d+2)} \left[\frac{1}{\mu^S + \mu} \left\{ \frac{1}{y} + O(1) \right\} + \frac{1}{\mu^C + \mu^S + \mu} \left\{ \frac{1}{6-y} + O(1) \right\} \right]. \tag{27}$$

Now writing $\mu^C/\mu^S = \alpha$, and using the Kolmogorov value $y=4$, we get after dividing Eq. (27) by Eq. (26)

$$3\alpha^2 + 4\alpha = \frac{d-1}{2}(\alpha^2 + 7\alpha + 6) \tag{28}$$

in the lowest order.

Thus in three dimensions, we have

$$2\alpha^2 - 3\alpha - 6 = 0, \tag{29}$$

yielding

$$\alpha = \frac{3 \pm \sqrt{57}}{4} \tag{30}$$

giving $\alpha = 2.6375$, choosing the positive sign. Using this result in Eq. (26) yields $\mu^S = 0.3758g$. These results may be compared with Kraichnan's numerical (exact) results, $\alpha = 2.163$ and $\mu^S = 0.296g$.

In two dimensions, on the other hand, we have

$$5\alpha^2 + \alpha - 6 = 0, \tag{31}$$

yielding

$$\alpha = \frac{-1 \pm 11}{10}, \tag{32}$$

so that $\alpha = 1$, exactly matching Kraichnan's result. Using this result in Eq. (27), we obtain $\mu^S = \sqrt{21/32}g = 0.8101g$, whereas Kraichnan obtained $\mu^S = 0.609g$.

Using the above results, the Kolmogorov constants can be evaluated, using Kraichnan's results [15]

$$C_{3D} = 3.022(\mu^S)^{2/3} = 1.5736g^{2/3} = 1.6401, \tag{33}$$

$$C_{2D} = 8.94(\mu^S)^{2/3} = 7.7689g^{2/3} = 8.09696, \tag{34}$$

and

$$\frac{C_{2D}}{C_{3D}} = 4.94, \tag{35}$$

in the lowest order of our perturbative evaluation.

V. DISCUSSION AND CONCLUSION

The main difficulty in evaluating the response integrals (7) and (8), exactly by analytical means, arises because they contain scale dependent relaxation rates as sums in the denominator of the integrands. The relaxation rates obey a scaling law, with fractional powers, leading to impossibility of evaluating these integrals exactly. Such integrals occur in critical dynamics of systems as varied as liquid helium, antiferromagnet, Heisenberg ferromagnet, etc., and various perturbative methods of evaluation of such integrals occuring in critical dynamics have been considered by Bhattacharjee and Ferrell [17].

We realize the importance of the application of the perturbative evaluation procedure in the case of the Kraichnan's test-field model of turbulence, because now we can compare the perturbative results with the exact ones. Kraichnan's exact results by numerical computation are $C_{3D} = 1.40$, $C_{2D} = 6.694$, and $C_{2D}/C_{3D} = 4.786$ [15]. When we compare them with the lowest-order perturbative results of our calculations presented in the last section [in Eqs. (33), (34), and (35)], we see that there is a good agreement between the results, keeping in mind the extent of approximation involved in the present evaluation procedure. It may be pointed out that in Kraichnan's (numerical) evaluation, the parameter y was held fixed at the Kolmogorov value of 4, whereas in the present perturbative scheme, we extracted the singularities at $y=0$ and $y=6$, and retained only the lowest-order terms. Considering this, we can say that the perturbative evaluation method yields reasonable estimates for the universal numbers.

The motivation for such perturbative evaluation by a double expansion about only two points can be justified when we see that a response integral, $I(y)$ say, is a function of y, and it has poles at several points $y = y_1, y_2, \ldots$. Thus, a Laurent expansion around all its poles can be constructed, yielding

$$I(y) = \frac{a_1}{y - y_1} + \frac{a_2}{y - y_2} + \cdots, \tag{36}$$

where $a_1, a_2, \ldots$ are the corresponding residues (strengths). The poles at $y=0$ and $y=6$ being the closest ones to the physical Kolmogorov value $y=4$, we expect that these poles will yield the maximum contribution.

We further point out an interesting feature of the perturbative results when we compare them with the existing results of renormalization group (RG) calculations in three and two dimensions. Yakhot and Orszag [18] used a Wilson type decimation scheme to carry out the RG calculations in three dimensions coupled with an ϵ expansion (see also Refs. [19,20]), leading to $C_{3D} = 1.6057$. Olla [21], on the other hand, used a field-theoretic RG formalism for the case of two-dimensional turbulence. Olla's result was shown to be consistent with a self-consistent formulation based on dynamic scaling ideas in Ref. [22]. It may be pointed out that Olla [21] used a definition of $E(k)$ different from that of Kraichnan [15]. A definition consistent with Kraichnan would introduce a factor of $\frac{1}{2}$ in Eq. (6) of Ref. [21]. Then Olla's (correct) result would be $C_{2D}=8.123$.

Thus we see that the RG results are in close agreement with our results of perturbative evaluations obtained in Eqs. (33) and (34). As we know, the RG schemes are based on ϵ expansion only about one point: about $y=0$ in three dimensions and about $y=6$ in two dimensions. However, when we look at the results of our perturbative evaluation, we find that

$$\mu^S=\frac{4g^2}{5\mu}\left[\frac{0.275}{y}+\frac{0.216}{6-y}\right], \tag{37}$$

$$\mu^C=\frac{8g^2}{5\mu}\left[\frac{0.5}{y}+\frac{0.216}{6-y}\right], \tag{38}$$

in three dimensions and

$$\mu^S=\mu^C=\frac{3g^2}{4\mu}\left[\frac{0.5}{y}+\frac{0.333}{6-y}\right] \tag{39}$$

in two dimensions. Thus we see the most interesting fact about the perturbative TFM result that both the UV and IR poles have strengths of the same order of magnitude. Consquently, the TFM closure is (approximately) equally sensitive to both the poles. This feature of the perturbative TFM results is in sharp contrast with the RG schemes, although the resulting Kolmogorov constants are in agreement with each other. This difference in pole structures in the two approaches (RG and TFM) is not unexpected because the underlying formulations of RG and TFM are entirely different. In both the Yakhot-Orszag RG and the RG of Olla, the starting points were the (forced) full Navier-Stokes equation. In contrast, Kraichnan's test-field model starts with the pressureless advection of a test field and subsequently the self-advection terms in the dynamics of the solenoidal and compressive parts are thrown away. Given the difference in the backbones of the two approaches (RG and TFM), it is worthwhile to discuss further about the origin of the differences between RG and TFM.

As noted above, the selective choice of interactions in TFM is made in order to gauge the effect of pressure. Consequently, TFM involves the geometrical factors $b^{S,C}(k,q,p)$ in the response equations (7) and (8), given by Eq. (9). As a result, the IR limit $p\ll k$ yields $O(p^2/k^2)$ term in the lowest contributing order, as can be seen in Eq. (24). It is, in fact, this (extra) factor of p^2/k^2 that makes TFM finite in the IR limit, and we now see that it contributes equally strongly as the UV pole.

This is at variance with the RG schemes which pick up contribution from only one of the poles depending on the space dimension. The Yakhot-Orszag RG in 3D recursively eliminates modes from the UV end using the full Navier-Stokes equation, with an imposed self-similarity of the recursive procedure. This allows an expansion only about $y=0$ (reminiscent of extraction of UV pole) and the IR behavior is irrelevant in this RG scheme. It may be noted in passing that Kraichnan's DIA closure, also based on the full Navier-Stokes equation (like RG), with no imposed selective choice of interaction (unlike TFM), involves a different geometrical factor $b(k,q,p)$ in its response equation. Consequently, the IR limit $p\ll k$ yields $O(p^0/k^0)$ term in the lowest contributing order, leading to IR catastrophe. In two dimensions, Olla's RG is based on the Navier-Stokes equation augmented with a drag term. In its field-theoretic setup with minimal subtraction, it extracts only the IR pole at $y=6$. The drag term in the dynamics plays an important role in this RG scheme.

It is because of the above differences in the underlying formulations (of RG and TFM) that RG is an expansion about only one value of y (0 or 6 depending on dimension) whereas, in perturbative TFM, both the poles are relevant; the difference in the pole structures is not therefore unexpected. However, what we would not have expected about the perturbative TFM is the surprising result that the two poles have roughly equal strengths in both three and two dimensions. And, to add more, the Kolmogorov constants following from the perturbative TFM agree well with the results of RG.

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